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Inflation centres of the cut and project quasicrystals

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Abstract. A rather general form of the conventional cut and project scheme is used to define quasicrystals as point sets in real n-dimensional Euclidean space. The inflation or, equivalently, the self-similarity properties of such quasicrystals are studied here assuming only the convexity of the acceptance window. Our result is a description of inflation centres of *all types* in a quasicrystal and a proof that our description is *complete*: there are no other inflation centres. For any chosen quasicrystal point ('internal inflation centre') u, its inflation properties are given as a set of scaling factors. It turns out that the scaling factors form a one-dimensional quasicrystal with a u-dependent acceptance window ('scaling window'). The intersection of the scaling windows associated with all points of a quasicrystal is the one-dimensional quasicrystal of universal ('internal') scaling symmetries. Its acceptance window is the interval $[0, 1]$. External inflation centres of a cut and project quasicrystal are those which are not among quasicrystal points. Their complete description is given analogically to the description of the internal ones imposing some additional requirements on the scaling factors. Between any two adjacent quasicrystal points one finds a countable infinity of external inflation centres. The scaling factors belonging to any such centre u form an infinite u-dependent subset of points of the quasicrystal with acceptance window containing $[0, 1]$.

1. Introduction

The importance of inflation symmetries in describing quasicrystals has long been recognized. Their role there is comparable with that of translation symmetries in the description of crystals. Many aspects and instances of such symmetries have been studied and are found in the literature; see for example [8, 20, 3, 1].

In this paper we study properties of infinite deterministic aperiodic point sets which share many properties with physical quasicrystals. For simplicity we speak of quasicrystals throughout the paper. In the absence of a generally accepted definition of quasicrystals, in this work we adopt as our definition a generic form of the cut and project method [19], taking a quasicrystal to be a point set in an n-dimensional Euclidean space $\mathbb{R}^n$. Moreover, we restrict our consideration to quasicrystals whose points have coordinates in the ring $\mathbb{Z}[\tau]$ of integers of the quadratic extension $\mathbb{Q}[\sqrt{5}]$ of rational numbers by $\sqrt{5}$. There are various possibilities of defining a quasicrystal, most of them can be (re)cast in terms of a certain cut and project scheme [14]. This method implies the existence of a region Ω, called an acceptance region (or acceptance window) and a mapping ('star map') under which all

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quasicrystal points are mapped into Ω. All quasicrystals here are assumed to be of the cut and project type, denoted by $\Sigma(\Omega)$, with the acceptance window Ω convex and closed.

Emergence of the cut and project method is usually associated with the work of de Bruijn [6] in connection with his study of the two-dimensional tilings of Penrose. Studies of its generalization to higher dimensions found a strong motivation after the discovery of materials with 'forbidden' symmetries whose x-ray spectra could be described using more than three integer coordinates. The earliest apparently independent extensions of the method appeared in [9, 2]. Subsequent exploitations of the method were numerous and fruitful. However, it was apparently only in [15, equation (5.1)] that a mathematical definition of a quasicrystal was formulated. The original definition was intended for quasicrystals with local symmetries of the type H_4 and of its subgroups. A generalization to any Coxeter group, crystallographic or not, or even unrelated to any local symmetry group, is taken here from [5]; it is also found in the lecture notes [19]. Unlike the previous cut and project algorithms, this is a mathematical definition which allows one to deduce readily a number of general (for example dimension-independent) properties of quasicrystalline point sets, such as aperiodicity, infinite repetition of any finite fragment, the inflation properties of this paper, description of the minimal distances in quasicrystals [10], and others. Its further generalization to quadratic irrationalities other than $\sqrt{5}$ is straightforward [7]. Several equivalent but very different methods of defining cut and project quasicrystals are found in [14]. They are based on the work of Meyer [12, 13].

The purpose of this paper is to rigorously study properties of quasicrystals in $\mathbb{R}^n$ which are usually called either self-similarity or inflation properties [3]. More precisely, we identify inflation symmetry centres and determine the scaling factors appropriate for each of them. It turns out that every quasicrystal point is a self-similarity (or inflation) centre of its quasicrystal and that inflation centres of $\Sigma(\Omega)$ are equally plentiful outside of the quasicrystal point set.

An inflation centre can be understood as the centre of a rescaling symmetry of the quasicrystal. The scaling factor determines the type of the symmetry. A point u in $\mathbb{R}^n$ is called an inflation centre of type $s \in \mathbb{R}$, or simply an s-inflation centre, of a quasicrystal $\Sigma(\Omega)$, if

$$s \cdot (y-u) + u \in \Sigma(\Omega) \qquad \text{for all } y \in \Sigma(\Omega). \tag{1}$$

It is useful to distinguish u in (1) according to whether $u \in \Sigma(\Omega)$ (*internal inflation centre*) or $u \notin \Sigma(\Omega)$ (*external inflation centre*). Definition (1) turns out to be a very fruitful generalization of the one given in [19] because it allows us to determine all scaling symmetries of $\Sigma(\Omega)$.

First we show that one needs to consider only scaling factors $s \in \mathbb{Z}[\tau]$ and points u having coordinates in $\mathbb{Q}[\sqrt{5}]$. Inflation centres u with coordinates in the ring $\mathbb{Z}[\tau]$ belong necessarily to the quasicrystal, otherwise u is an external inflation centre.

In theorem 3.2 we provide: (i) a description of internal inflation centres of *all types*, and (ii) a proof that our description is *complete*: there are no other inflation centres of any type among the quasicrystal points. It turns out that the scaling factors s associated with a point $u \in \Sigma(\Omega)$ form a one-dimensional quasicrystal whose acceptance window depends on u. Considering all the points u of $\Sigma(\Omega)$ and their corresponding u-dependent acceptance windows at once, it is natural to ask about the intersection of these windows. It turns out to be an interval independent of Ω, namely $[0, 1]$.

Perhaps the most interesting is the universality of the implication of this result. All points of any quasicrystal $\Sigma(\Omega) \subset \mathbb{R}^n$ with practically any closed acceptance window (assuming only the convexity of Ω) in any dimension $n \geqslant 1$, are inflation symmetry centres with surprisingly many scaling factors s in common, namely all $s \in \Sigma([0, 1])$.

The description of external inflation centres requires some generalization of the proof of theorem 3.2, which leads to theorem 3.5. For a fixed external inflation centre u the scaling factors form a rescaled one-dimensional quasicrystal. It is the intersection of an ideal in $\mathbb{Z}[\tau]$, shifted by 1, with a one-dimensional quasicrystal given by the same expression as for the internal inflation centres in theorem 3.2.

2. Mathematical preliminaries

Consider the two roots of the algebraic equation $x^2 = x + 1$, the well known golden ratio τ and its conjugate τ'. In the extension of the rational numbers by τ, $\mathbb{Q}[\tau] = \{s + t\tau | s, t \in \mathbb{Q}\}$, there is an automorphism $' : \mathbb{Q}[\tau] \to \mathbb{Q}[\tau]$ given by $\tau \to \tau'$. The ring $\mathbb{Z}[\tau]$ of integers of $\mathbb{Q}[\tau]$ is the set $\mathbb{Z}[\tau] = \{a + b\tau | a, b \in \mathbb{Z}\}$. The ring $\mathbb{Z}[\tau]$ is a ring of principal ideals; they are of the form $\zeta\mathbb{Z}[\tau]$, for a $\zeta \in \mathbb{Z}[\tau]$. In particular, $\mathbb{Z}[\tau]$ it is a unique factorization domain, which assures that the greatest common divisor and the lowest common multiple are well defined. For any $u \in \mathbb{Q}[\tau]$, there exist $p, q \in \mathbb{Z}[\tau]$, $\gcd\{p, q\} = 1$, such that $u = \frac{p}{q}$. The set $\{\pm\tau^k | k \in \mathbb{Z}\}$ is the group of units of $\mathbb{Z}[\tau]$.

The stage for building an n-dimensional quasicrystal in a Euclidean space $\mathbb{R}^n$ is the torsion free $\mathbb{Z}[\tau]$-module $M := \sum_{i=1}^{n} \mathbb{Z}[\tau]\alpha_i$, with the basis $\alpha_i, \dots, i = 1, .., n$. $M \subset \mathbb{R}^n$ is called a $\mathbb{Z}[\tau]$-lattice of rank n, if it spans $\mathbb{R}^n$ over $\mathbb{R}$. It is an everywhere dense set of points. We consider $\mathbb{Z}[\tau]$-lattices for which the standard scalar product of lattice vectors takes values in $\mathbb{Q}[\tau]$.

Let $*$ be a mapping $* : M \to M^* \subset \mathbb{R}^n$. It is called a 'star map' if, for any $x, y \in M$ and any $r \in \mathbb{Z}[\tau]$, it is: (i) additive: $(x + y)^* = x^* + y^*$; (ii) semilinear: $(rx)^* = r'x^*$, and (iii) M^* spans $\mathbb{R}^n$. The star map is uniquely extended to the rational span $\mathbb{Q}M$ of M. Note that if the scalar product of vectors $x, y \in M$ is $\mathbb{Z}[\tau]$-valued, then $(x|y) = (x^*|y^*)'$.

As a consequence of these properties, M^* is a dense point set in $\mathbb{R}^n$. The criterion for choosing quasicrystal points is whether or not the star map image of the points falls into a chosen acceptance window $\Omega \subset \mathbb{R}^n$.

Definition 2.1. Let M be a $\mathbb{Z}[\tau]$-lattice in $\mathbb{R}^n$ and let Ω be a bounded convex and closed subset of $\mathbb{R}^n$. The set of points $\Sigma(\Omega)$,

$$\Sigma(\Omega) = \{x \in M | x^* \in \Omega\} \tag{2}$$

is a cut and project quasicrystal; Ω is called an acceptance window for $\Sigma(\Omega)$.

In general, no further conditions on Ω have to be imposed. The boundedness of the acceptance window assures the Delaunay property of $\Sigma(\Omega)$ [14], the convexity is important for τ-inflation invariance [4], and the closure of Ω simplifies the proof of one of our statements. Partially open or open Ω would complicate the argument. Details of an $(n-1)$-dimensional boundary of Ω could not influence any conclusions pertinent to physics. Indeed, from a viewpoint of a physicist all real-life quasicrystals are necessarily of finite size, containing a finite number of points—atoms. Consequently, Ω is a discrete set of points, not a dense one! In fact it is another quasicrystal.

We call the quasicrystals defined in (2) of the cut and project type. However, the definition does not make obvious the location of the crystallographic lattice in which the 'cutting' takes place and from which the 'projecting' is done. Instead we have been operating exclusively with the points of M and M^*.

In [5, 19] it was shown that (M, M^*) can be viewed as a *crystallographic lattice* $\tilde{M}$ in $\mathbb{R}^{2n}$, i.e. the scalar product of any two vectors of the lattice is integer. To see this, take any $x \in M$ and consider x and x^* as a single point $(x, x^*) \in \tilde{M} \subset \mathbb{R}^{2n}$. The integer-valued scalar product is defined in $\tilde{M}$ by the scalar products in M and M^* which both take values in $\mathbb{Z}[\tau]$:

$$((x, x^*)|(y, y^*)) := \frac{(x|y)}{2+\tau} + \frac{(x^*|y^*)}{2+\tau'} \qquad \forall x, y \in M. \tag{3}$$

In particular, for $n = 1$, it is easy to verify directly that, for any (x, x^*) with $x = a + b\tau$, $a, b \in \mathbb{Z}$, we can write $(a + b\tau, a + b\tau') = a(1, 1) + b(\tau, \tau')$ and that $(1, 1)$ and (τ, τ') form an orthonormal basis of the two-dimensional square lattice $\tilde{M}$. Indeed,

$$((1, 1)|(1, 1)) = 1 \qquad ((\tau, \tau')|(\tau, \tau')) = 1 \qquad ((1, 1)|(\tau, \tau')) = 0.$$

Then M and M^* are two projections of points of $\tilde{M}$ on subspaces orthogonal to each other with respect to (3). The acceptance window $\Omega \subset M^*$ determines the 'cut' which is being made in $\tilde{M}$. The quasicrystal $\Sigma(\Omega)$ then consists of the points which are projected from the 'cut' to M.

Consider an example of (2): a one-dimensional quasicrystal. In this case Ω is an interval in $\mathbb{R}$ and quasicrystal points are numbers from $\mathbb{Z}[\tau]$. A star map on $\mathbb{Z}[\tau]$ is the automorphism $'$,

$$(a + \tau b)^* = a + \tau' b \qquad a, b \in \mathbb{Z}.$$

We choose $\Omega = [0, 1]$, the acceptance region of the quasicrystal $\Sigma([0, 1])$, which plays an important role in this paper. One verifies directly from definition (2), that the first few points nearest to 0 in the quasicrystal $\Sigma([0, 1])$ are the following

$$\begin{aligned} &\ldots, -\tau, 0, 1, 1+\tau, 2+2\tau, 2+3\tau, 3+4\tau, 4+5\tau, 4+6\tau, \\ &\qquad 5+7\tau, 5+8\tau, 6+9\tau, 7+10\tau, \ldots \end{aligned} \tag{4}$$

Note that the quasicrystal $\Sigma([0, 1])$ has a global reflection symmetry in the point $\frac{1}{2}$ and therefore its negative points are readily obtained from those greater than 0. Indeed, for every $x \in \Sigma([0, 1])$, one has $y = 1 - x \in \Sigma([0, 1])$ because $x^* \in [0, 1] \Leftrightarrow y^* = 1 - x^* \in [0, 1]$.

Let us consider the distances between adjacent points of $\Sigma([0, 1])$ as tiles. It is curious to notice that $\Sigma([0, 1])$ has an exceptional tile which occurs precisely once, namely the one between the points 0 and 1. All other tiles of $\Sigma([0, 1])$ have the length either τ or τ^2. In general, one-dimensional subquasicrystal of any cut and project quasicrystal $\Sigma(\Omega) \subset \mathbb{R}^n$ has either two or three tiles. Demonstrations and exact formulation of these phenomena are presented in [10].

Examples of two-dimensional quasicrystals, calculated using (2), are found in many places, for example in [15–19, 5].

3. Internal and external inflation centres

In this section we address the rescaling invariances of a quasicrystal with respect to various fixed points in $\mathbb{R}^n$.

The first example of inflation symmetry of this kind is the rescaling of any $x \in \Sigma(\Omega)$ by τ^k, $k \in \mathbb{N}$, in the case of an acceptance window Ω centred at origin, centrally symmetric, and convex. The inflation centre here is the origin. More general is the definition of an inflation centre u of degree $d \in \mathbb{N}$,

$$\tau^d \cdot (y - u) + u \in \Sigma(\Omega) \qquad \forall y \in \Sigma(\Omega). \tag{5}$$

Some examples of inflation properties (5) of the internal type are found in [19]. Our definition (1) of an inflation centre includes (5) as a special case. The inflation centres which are external to the set $\Sigma(\Omega)$ have apparently not been exemplified in the literature before. Here we describe them all.

Note that definition (1) imposes restrictions neither on the inflation centre $u \in \mathbb{R}^n$, nor on the scaling factor $s \in \mathbb{R}$. Let us first determine which u's and s's may give a nontrivial scaling symmetry of a cut and project quasicrystal.

Suppose that $u \in \mathbb{R}^n$ is an s-inflation centre for some $s \in \mathbb{R}$. For any two quasicrystal points y_1 and y_2, one has $s(y_1 - y_2) = (u + s(y_1 - u)) - (u + s(y_2 - u)) \in \Sigma(\Omega) - \Sigma(\Omega) \subset M$. Due to definition (2) of $\Sigma(\Omega) \subset M$, there is no proper $\mathbb{Z}[\tau]$-submodule of M containing $\Sigma(\Omega)$, therefore s belongs to $\mathbb{Z}[\tau]$ (see also [1]). For fixed $s \in \mathbb{Z}[\tau]$, the condition $(1 - s)u + sy \in \Sigma(\Omega) \subset M$, for all $y \in \Sigma(\Omega)$, implies that u is an element of the rational span $\mathbb{Q}M$ of the $\mathbb{Z}[\tau]$-module M.

For our purpose it is convenient to reformulate definition (1) in an equivalent way.

Remark 3.1. Let $s \in \mathbb{Z}[\tau]$ and $\Sigma(\Omega) \subset \mathbb{R}^n$ be a cut and project quasicrystal. Definition (1) is equivalent to the statement: $u + x \in \Sigma(\Omega)$ implies $u + sx \in \Sigma(\Omega)$, i.e. u is an s-inflation centre if and only if

$$u + x \in \Sigma(\Omega) \Longrightarrow u + sx \in \Sigma(\Omega).$$

In this paper we provide a description of all internal and external inflation centres and all corresponding scaling factors. For $u \in \mathbb{Q}M$ the star map is well defined. A point u such that u^* does not belong to the closure $\overline{\Omega}$ of Ω, cannot be a nontrivial inflation centre (see remark 3.1).

Suppose first that u is an element of the $\mathbb{Z}[\tau]$-module M. Assuming that Ω is closed, i.e. $\overline{\Omega} = \Omega$, $u^* \in \Omega$ implies that $u \in \Sigma(\Omega)$. A complete description of the internal inflation centres (1) is provided in theorem 3.2. The external inflation centres of $\Sigma(\Omega)$ are described by theorem 3.5.

Theorem 3.2. Let $\Sigma(\Omega) \subset \mathbb{R}^n$ be a quasicrystal with bounded closed convex acceptance region $\Omega \subset \mathbb{R}^n$. For any $u \in \Sigma(\Omega)$ we denote by $\sigma(x, u)$, $\rho(x, u)$, and $\mu(u)$ the following:

$$\rho(x, u) = \inf\{\|(kx)^*\| \mid (u + kx)^* \in \mathbb{R}^n \setminus \Omega; k \in \mathbb{Z}[\tau]\} \tag{6}$$

$$\sigma(x, u) = \sup\{\|(kx)^*\| \mid (u + kx)^* \in \Omega; k \in \mathbb{Z}[\tau]\} \tag{7}$$

$$\mu(u) = \inf\left\{ \frac{\rho(x, u)}{\sigma(x, u)} \middle| (u + x)^* \in \Omega \right\}. \tag{8}$$

A point $u \in \Sigma(\Omega)$ is an s-inflation centre if and only if s belongs to the one-dimensional quasicrystal $\Sigma([-\mu(u), 1])$.

A proof of the theorem is based on the following two lemmas. In the first one we find a set of $s \in \mathbb{Z}[\tau]$ such that u is an internal s-inflation centre.

Lemma 3.3. Let $\Sigma(\Omega) \subset \mathbb{R}^n$ be a quasicrystal with bounded closed acceptance region $\Omega \subset \mathbb{R}^n$. If Ω is convex, then any point $u \in \Sigma(\Omega)$ is an s-inflation centre for any $s \in \Sigma([-\mu(u), 1])$, i.e.

$$s \cdot (y - u) + u \in \Sigma(\Omega) \qquad \forall u, y \in \Sigma(\Omega) \qquad \forall s \in \Sigma([-\mu(u), 1]). \tag{9}$$

Proof. We use remark 3.1. Clearly, $\Sigma([-\mu(u), 1]) = \Sigma([0, 1]) \cup \Sigma([-\mu(u), 0))$. Since $\Sigma([0, 1])$ is independent of u, it is convenient to study the two cases separately.

Take any $u \in \Sigma(\Omega)$, i.e. $u^* \in \Omega$. First, let us show that u is an $\Sigma([0, 1])$-inflation centre. Take any $s \in \Sigma([0, 1])$. If $y \equiv u + x \in \Sigma(\Omega)$ then $y^* \in \Omega$. Since $s \in \Sigma([0, 1])$, we have $0 \leqslant s' \leqslant 1$, so that the point $(u + sx)^* = u^* + s'x^*$ lies on the straight line between u^* and

$y^* = (u+x)^*$. (Note that if $u, y \in M$ then $x = y-u \in M$, i.e. the symbol x^* is meaningful.) However, $u^*, y^* \in \Omega$ and Ω is convex, so that also $(u+sx)^* \in \Omega \Longleftrightarrow u+sx \in \Sigma(\Omega)$.

Next consider $s \in \Sigma([-\mu(u), 0))$. The steps are similar. It is only necessary to show the implication

$$(u+x)^* \in \Omega \Longrightarrow (u+sx)^* \in \Omega. \tag{10}$$

However, the number $\mu(u)$ was defined precisely to assure that implication. Indeed, take any $y^* = (u+x)^* \in \Omega$. The point $(u+sx)^*$ lies on the straight line determined by u^* and y^*. Since $-\mu(u) \leqslant s' < 0$, we have $|s'| = -s' \leqslant \frac{\rho(x,u)}{\sigma(x,u)}$ for any $x \in M$ such that $y^* \equiv (u+x)^* \in \Omega$.

Recall the definition of $\rho(x,u)$ and $\sigma(x,u)$. The points u^* and $y^* \equiv (u+x)^*$ determine a straight line in $\mathbb{R}^n$, which has a nonempty convex intersection with Ω, which we denote by p_x. The functions $\rho(x,u)$ and $\sigma(x,u)$ are the minimal and maximal distances between u^* and some point in p_x. We have

$$y^* = (u+x)^* \in p_x \Longrightarrow \|y^* - u^*\| = \|x^*\| \leqslant \sigma(x,u) \tag{11}$$

and

$$y^* = (u+x)^* \in p_x \Longleftarrow \|y^* - u^*\| = \|x^*\| \leqslant \rho(x,u). \tag{12}$$

The latter holds even with the nonsharp inequality since Ω is closed, i.e. the infimum (cf the definition of $\rho(x,u)$) is not achieved on points $(u+x)^* \notin \Omega$.

Using (11), we obtain

$$\|(u+sx)^* - u^*\| = |s'| \cdot \|x^*\| \leqslant |s'| \cdot \sigma(x,u) \leqslant \frac{\rho(x,u)}{\sigma(x,u)} \cdot \sigma(x,u) = \rho(x,u).$$

Therefore

$$|s'| \cdot \|x^*\| \leqslant \rho(x,u).$$

Due to (12), $(u+sx)^* \in p_x$, i.e. $(u+sx)^* \in \Omega$. $\square$

The quasicrystal $\Sigma([0,1])$ is a common subset of all $\Sigma([-\mu(u),1])$. It is independent of Ω and also the dimension of the space.

This observation is far reaching: all points of all quasicrystals $\Sigma(\Omega) \subset M \subset \mathbb{R}^n$ with all convex acceptance windows in all dimensions $n \geqslant 1$, are internal inflation symmetry centres with surprisingly many scaling factors s in common, namely all $s \in \Sigma([0,1])$. However, each $u \in \Sigma(\Omega)$ has other scaling factors as well as those of $\Sigma([0,1])$, forming the u-dependent quasicrystal $\Sigma([-\mu(u),0))$.

Let us consider an example of a H_2 cut and project quasicrystal $\Sigma(\Omega)$, whose acceptance window Ω is a disk (of radius 5), and let us illustrate the abundance of its internal inflation symmetries given by various points of $\Sigma([0,1])$. One of its external inflation centres is pointed out at the end of this section together with its scaling factors.

We say that a quasicrystal is of H_2-type if M is a linear combinations of simple roots of H_2 with coefficients from $\mathbb{Z}[\tau]$. The Coxeter group H_2 has two simple roots. These are equal length vectors, the angle between them is $4\pi/5$. A standard model [19, 5] of the simple roots in the complex plane is 1 and $\xi = \mathrm{e}^{\frac{4\pi \mathrm{i}}{5}}$. Their star map is then $1^* = 1$ and $\xi^* = \xi^2$. In figure 1 a circular window view of such a quasicrystal is presented. Four sets of examples of (1) are shown in which the inflation factor s takes the values (4) from $\Sigma([0,1])$ and u is a quasicrystal point.

The following lemma shows that no point $u \in \Sigma(\Omega)$ has other scaling symmetries.

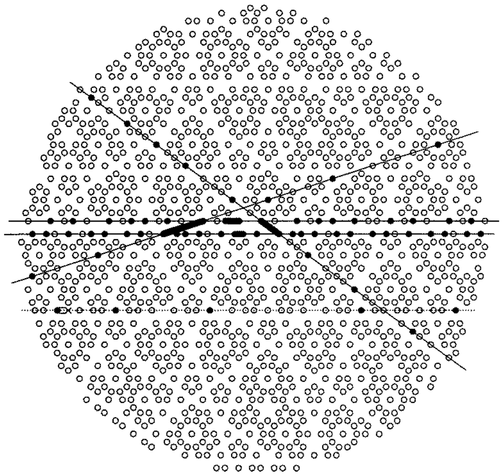

Figure 1. A disk-shaped fragment of a cut and project quasicrystal with a circular acceptance window is shown. Small circles represent the quasicrystal points. The four full light lines across the quasicrystal indicate four sets of examples of internal inflation symmerties with the scaling factors from $\Sigma([0,1])$. The endpoints of the thick segment on a line are the points chosen to represent u and y in (1). (The two points are interchangable due to the global symmetry of $\Sigma([0,1])$ around the point $\frac{1}{2}$.) The full circles are the points scaled according to (1). The values of the scaling factor s which were used are the nearest points to 0 in $\Sigma([0,1])$. Most of them are listed in (4). The dotted line indicates a similar example of an external inflation symmetry centre denoted by the cross. The distance, which is scaled, is the one between the cross and the full circle nearest to it. The scaling factors are from (17).

Lemma 3.4. Let $\Sigma(\Omega) \subset \mathbb{R}^n$ be a quasicrystal with bounded acceptance region $\Omega \subset \mathbb{R}^n$, and let $s \in \mathbb{Z}[\tau]$. If u is an internal s-inflation centre of $\Sigma(\Omega)$, then $s \in \Sigma([-\mu(u), 1])$.

Proof. We prove this by contradiction. Suppose there exists an $s \notin \Sigma([-\mu(u), 1])$ such that u is the s-inflation centre of $\Sigma(\Omega)$. We find an $y^* = (u+x)^* \in \Omega$, such that $(u+sx)^* \notin \Omega$. Since $s \notin \Sigma([-\mu(u), 1])$, either $s' > 1$ or $s' < -\mu(u)$.

Let us first consider the case $s' > 1$. Since the image of $\Sigma(\Omega)$ under the star map is dense in Ω, we can find a point $y^* = (x+u)^* \in \Omega$ close enough to the boundary of Ω, such that $(u+sx)^*$ falls out of the acceptance domain. Thus the necessary condition for u to be an s-inflation centre is not fulfilled, which is a contradiction.

If $s' < -\mu(u)$ then from the infimum property of $\mu(u)$, there exists an $x \in M$, such that $(u+x)^* \in \Omega$ and $|s'| = -s' > \frac{\rho(x,u)}{\sigma(x,u)}$.

At first, let us discuss the case $\rho(x,u) < \sigma(x,u)$. Note that for any kx, such that $\rho(x,u) < \|(kx)^*\| < \sigma(x,u)$, exactly one of the points $(u+kx)^*$, $(u-kx)^*$ belongs to Ω. Consider ε such that

$$0 < \varepsilon < \min\left\{|s'| - \frac{\rho(x,u)}{\sigma(x,u)}; 1 - \frac{\rho(x,u)}{\sigma(x,u)}\right\}. \tag{13}$$

From the supremum property of $\sigma(x,u)$ there exists k such that $(kx+u)^* \in \Omega$ and

$$\|(kx)^*\| > \sigma(x,u) \cdot \frac{\rho(x,u)}{\rho(x,u) + \varepsilon\sigma(x,u)} > \rho(x,u). \tag{14}$$

Note that (14) forces $(u-kx)^* \notin \Omega$.

Since we suppose that u is an s-inflation centre, we have $u + s(kx) \in \Sigma(\Omega)$ and thus $(u + skx)^* \in \Omega$. According to the definition of $\rho(x, u)$ and since $s' < 0$, it means that

$$\|(skx)^*\| \leqslant \rho(x, u). \tag{15}$$

Combining (14) and (15) we obtain

$$|s'| = \frac{\|(skx)^*\|}{\|(kx)^*\|} < \rho(x, u) \cdot \frac{\rho(x, u) + \varepsilon\sigma(x, u)}{\sigma(x, u)\rho(x, u)} = \frac{\rho(x, u)}{\sigma(x, u)} + \varepsilon$$

and thus a contradiction with (13).

In the case when $\rho(x, u) = \sigma(x, u)$ is $|s'| = -s' > \frac{\rho(x,u)}{\sigma(x,u)} = 1$ and clearly for a point $(u + kx)^* \in \Omega$ sufficiently close to the boundary of Ω, the point $(u + skx)^* \notin \Omega$, thus u is not an s-inflation centre. □

So far we have described all the inflation centres of a quasicrystal $\Sigma(\Omega)$ that belong to the $\mathbb{Z}[\tau]$-module M and hence also to $\Sigma(\Omega)$ (internal inflation centres). We have pointed out that any candidates for nontrivial external inflation centres must be elements of $\mathbb{Q}M$, such that $u^* \in \Omega$. A description of the external inflation centres can be easily derived from theorem 3.2. Carrying out similar considerations for fixed $u \in \mathbb{Q}M$ as in the proofs of lemmas 3.3 and 3.4, one has to answer the question for which factors $s \in \Sigma([-\mu(u), 1])$, the fact that $u + x \in M$ implies $u + sx = (1 - s)u + s(u + x) \in M$. It turns out that all scaling factors have to belong to the set $(1 - \zeta\mathbb{Z}[\tau])$, corresponding to certain ideal $\zeta\mathbb{Z}[\tau]$ of $\mathbb{Z}[\tau]$, namely for ζ, such that $\zeta u \in M$ and that there is no proper $\mathbb{Z}[\tau]$-submodule L of M containing ζu. The factor ζ can be found as the lowest common multiple of denomainators of coordinates of u in the basis α_i of the $\mathbb{Z}[\tau]$-module M. More precisely, considering the notation of theorem 3.2, one has theorem 3.5, where all the inflation centres of $\Sigma(\Omega)$, both internal and external, are described.

Theorem 3.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex and closed region. Let $u \in \mathbb{Q}M$, $u^* \in \Omega$,

$$u = \sum_{i=1}^{n} \frac{p_i}{q_i}\alpha_i$$

where α_i, $i = 1, \ldots, n$, is the basis of the $\mathbb{Z}[\tau]$-module M, and p_i, q_i are numbers in $\mathbb{Z}[\tau]$, such that $\gcd\{p_i, q_i\} = 1$. Denote by ζ the lowest common multiple of q_i, $i = 1, \ldots, n$. Then u is an s-inflation centre of $\Sigma(\Omega)$, if and only if

$$s \in \Sigma([-\mu(u), 1]) \cap (1 - \zeta\mathbb{Z}[\tau]). \tag{16}$$

Let us now illustrate the presence of the external inflation centre on the quasicrystal in figure 1, according to theorem 3.5. Between any two adjacent quasicrystal points one finds a countable infinity of external inflation centres. The scaling factors belonging to any such centre u form an infinite u-dependent subset of points of $\Sigma([-\mu(u), 1]) \supset \Sigma([0, 1])$.

We choose the inflation centre u so that $\zeta = 2$. We select the points $s \in \Sigma([0, 1])$ from (4), which are divisible by 2. The scaling factors, appropriate to u, are of the form $1 - s$. They satisfy (16), since $1 - \Sigma([0, 1]) = \Sigma([0, 1])$. The scaling factors used in the exapmle of external inflation centre are

$$1, \; -1 - 2\tau, \; -3 - 6\tau, \; -7 - 12\tau, \; -9 - 16\tau. \tag{17}$$

4. Concluding remarks

(1) The quasi-addition operation on quasicrystal points introduced by Berman and Moody [4], which has occasionally been called τ-inflation [5], can be understood as a special case of (1). Indeed, it follows from two facts, namely that: (i) the scaling properties given by the elements of $\Sigma([0,1])$ apply to all points of $\Sigma(\Omega)$, and (ii) $s=-\tau$ is among the elements of $\Sigma([0,1])$. Explicitly, we have the equality of operations

$$\tau^2 x - \tau y = -\tau(y-x) + x$$

where on the left is the quasi-addition $x \vdash y$ and on the right is the $(-\tau)$-inflation.

Consequently, there is a nontrivial quasiaddition-like operation on $\Sigma(\Omega)$ for every element of $\Sigma((0,1))$. Practically, the most interesting are the lowest values shown in (4). By taking an open interval $(0,1)$ instead of $[0,1]$, we are excluding only the trivial possibilities $s=0$ and $s=1$. For example, put $s=2\tau^2$. Then we define the operation

$$x \vdash y := 2\tau^2(y-x) + x = 2\tau^2 y - (1+2\tau)x = 2\tau^2 y - \tau^3 x \qquad \forall x, y \in \Sigma(\Omega)$$

under which $\Sigma(\Omega)$ is closed by theorem 3.2.

(2) Our assumption that Ω is closed in $\mathbb{R}^n$ is not very essential for most of our conclusions. It greatly simplifies the arguments. Open or partially closed Ω would, for example, remove at most one element from the quasicrystal of scaling factors, namely $-\mu'(u)$, if it is in $\mathbb{Z}[\tau]$ and satisfies some further conditions. Also, one would find nontrivial external inflation centres inside the $\mathbb{Z}[\tau]$-module, namely those $u \in M$ which under star maps are mapped into the boundary of Ω.

(3) It is shown in [11] that for any $(-\tau)$-inflation invariant Delaunay point set in $\mathbb{R}^n$, one can find an affine mapping into a $\mathbb{Z}[\tau]$-module M. The resulting point set is a cut and project quasicrystal whose acceptance region has a convex interior. Hence, results of this paper are also applicable to such cases.

(4) It is well known that some special two-dimensional quasicrystals can be constructed using self-similarity of the corresponding tilings. It would be interesting to investigate the limits of the possibilities of constructing quasicrystals by means of their s-inflation properties.

It is likely that here one has a tool for dealing with a much larger class of quasicrystals and which is not limited to two dimensions.

(5) There is an aspect of the theory of quasicrystalline point sets which is evidently waiting to be fully investigated, namely its connection with the theory of fractals. It has long been recognized that the map $M \to M^*$ is a fractal-like contraction map: an inflation

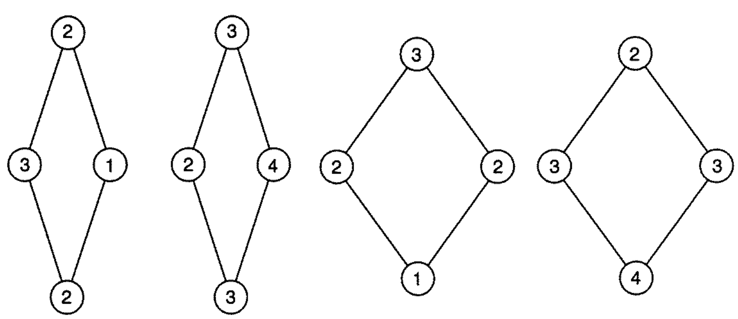

Figure 2. Matching rules given by four-colouring of the vertices of Penrose rhombs.

transformation in M is a contraction in M^*. Therefore the completeness of our description of the inflation symmetries can be considered as a step in clarifying that connection.

(6) Undoubtedly the best known and by far the most studied quasicrystals arise as the two-tile tilings of the plane of Penrose [6]. The vertices of the tiles form a quasicrystal in $\mathbb{R}^2$ to which the present results do *not* apply, because their acceptance region does not satisfy the convexity requirement. However, the quasicrystal of Penrose can be viewed as a union of four subquasicrystals with four different acceptance regions. In the case of rhombic Penrose tiling, there are four different pentagonal acceptance regions. Hence, our results are applicable to each of the four subquasicrystals separately but not to their union. For details of such a view on Penrose tilings, see section 4 in [17].

Curiously, during such a chromatic decomposition of Penrose quasicrystals, the well known matching rules for the rhombes are translated into the requirement to match the four colours of the vertices of the tiles shown in figure 2. The colour numbers are given by (4.6) and (3.12) in [17] with $s_1 = s_2 = 1$.

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